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FACULTY OF MATHEMATICS

CONTROLLED STOCHASTIC DIFFERENTIAL EQUATIONS WITH DELAY

Summary of the PhD thesis

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IAŞI - 2013

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You are kindly invited to participate to the public defense of the PhD thesis.

Contents

1 Introduction **1**

2 Preliminaries **4**

2.1 Elements of nonlinear analysis 4

2.2 Elements of stochastic analysis 4

3 Multivalued stochastic differential equations with delay **5**

3.1 Assumptions 6

3.2 A priori estimates 7

3.3 Existence and uniqueness 8

4 Multivalued BSDEs with time-delayed generator **9**

4.1 Assumptions 10

4.2 Existence and uniqueness 12

5 Stochastic optimal control **13**

5.1 Dynamic programming principle 13

5.1.1 Problem formulation 13

5.1.2 Properties of the value function 15

5.1.3 Hamilton-Jacobi-Bellman Equation. Viscosity solution 18

5.2 Necessary conditions of optimality 20

5.2.1 Statement of the problem 20

5.2.2 Variation equation 25

5.2.3 Maximum principle for near optimal controls 28

5.2.4 Maximum principle 29

References **30**

Chapter 1

Introduction

In most stochastic modeling of random phenomena one assumes that the future state depends only on the present state and not on the past history, and furthermore, that the influence of the present state is instantaneous. However, for some systems this assumption doesn't hold true. In order to describe the behavior of such systems, stochastic models with delay have been used. The advantage of explicitly incorporating time delays in modeling is to recognize the reality of non-instantaneous interactions. Delayed stochastic models appear in various applications in engineering where the dynamics are subject to propagation delay. For instance, in financial models where the past history of a stock price has an effect in the determination of the fair price of a call option, in marketing models where there is a time lag between advertising expenditures and the corresponding effect on the goodwill level.
In this thesis we are interested in the study of forward and backward delayed stochastic differential equations with constraints on the state. In this framework we consider the stochastic delayed differential equation of multivalued type, also called stochastic delayed variational inequality (where the solution is forced, due to the presence of term $\partial\varphi\left(X(t)\right)$, to remain into the convex set $\overline{\mathrm{Dom}\left(\varphi\right)}$). Still in the framework of delayed stochastic differential equations with state constraints, we are also interested in a new type of multivalued backward stochastic differential equation (BSDE) with time-delayed generator.

Beside the problem of existence and uniqueness of a solution of the forward and backward multivalued stochastic differential equation with time delay, we study the stochastic control problem. We first show that the value function satisfies the dynamic programming principle, then it is proved that the value function is a viscosity solution of a proper Hamilton-Jacobi-Bellman equation. We also derive the Pontryagin maximum principle; in this regard we start by the derivation of the variation equation, then we introduce the maximum principle for near-optimal controls and finally, we give the necessary conditions of optimality.

This thesis is organized as follow :

In Chapter 2, we recall some important results of nonlinear analysis and stochastic analysis.

In Chapter 3, we study the following multivalued stochastic differential equation with delay (also called the stochastic delay variational inequality) :

$$\begin{cases} dX(t) + \partial\varphi\left(X(t)\right)dt \ni b\left(t, X(t), \Theta(t), \Pi(t)\right)dt \\ \qquad\qquad\qquad\qquad + \sigma\left(t, X(t), \Theta(t), \Pi(t)\right)dW(t),\ t \in (0,T], \\ X(t) = \xi\left(t\right),\ t \in \left[-\delta, 0\right], \end{cases} \tag{1.1}$$

where $\delta \geq 0$ is the fixed delay, $\xi \in C\big(\left[-\delta, 0\right]; \overline{\mathrm{Dom}\left(\varphi\right)}\big)$, b and σ are given functions, φ is a convex lower semicontinuous function with $\partial\varphi$ its subdifferential, and Π, Θ are defined as follow

$$\Pi(t) := X(t-\delta), \quad \Theta(t) := \int_{-\delta}^{0} e^{\lambda r} X(t+r)dr = e^{-\lambda t}\int_{t-\delta}^{t} e^{\lambda s} X(s)\,ds) \tag{1.2}$$

In the first part of the chapter we provide the assumptions and some a priori estimates of the solution are also given. The last part concerns the existence and uniqueness theorem. The proof is based on the penalization method, by approximating φ by its Moreau-Yosida regularization.

In Chapter 4, we consider the following multivalued backward stochastic differential equation with time-delayed generator :

$$\begin{cases} -dY\left(t\right) + \partial\varphi\left(Y\left(t\right)\right)dt \ni F\left(t, Y\left(t\right), Z\left(t\right), Y_t, Z_t\right)dt \\ \qquad\qquad\qquad\qquad + Z\left(t\right)dW\left(t\right),\ 0 \leq t \leq T, \\ Y\left(T\right) = \xi\ . \end{cases} \tag{1.3}$$

We mention that in (1.3) the generator F at the moment $t \in [0,T]$ depends on the past values (Y_t, Z_t) on $[0,t]$ of the solution $\left(Y\left(t\right), Z\left(t\right)\right)$, where

$$Y_t := \left(Y(t+\theta)\right)_{\theta\in[-T,0]} \quad \text{and} \quad Z_t := \left(Z(t+\theta)\right)_{\theta\in[-T,0]}\ . \tag{1.4}$$

We set the assumptions and the definition of the solution for such BSDE. Also we prove the existence and uniqueness of the solution using the penalization method.

In Chapter 5, the stochastic optimal control problem is investigated. The first part of the chapter is devoted to the dynamic programming principle. The problem is to minimize the cost functional

$$J(s,\xi;u)=\mathbb{E}\Big[\int_s^T f\left(t,X(t),\Theta(t),u(t)\right)dt+h\left(X(T),\Theta(T)\right)\Big] \tag{1.5}$$

over a class of control strategies denoted by $\mathcal{U}\left[s,T\right]$ (here f and h are only continuous and with polynomial growth), subject to the following equation

$$\begin{cases} dX(t)+\partial\varphi\left(X(t)\right)dt\ni b\left(t,X(t),\Theta(t),\Pi(t),u(t)\right)dt \\ \qquad\qquad\qquad +\sigma\left(t,X(t),\Theta(t),\Pi(t),u(t)\right)dW(t),\ t\in(s,T], \\ X(t)=\xi\left(t-s\right),\ t\in\left[s-\delta,s\right], \end{cases} \tag{1.6}$$

We define the value function

$$\begin{aligned} &V\left(s,\xi\right)=\inf_{u\in\mathcal{U}[s,T]}J(s,\xi;u),\ (s,\xi)\in[0,T)\times C\big(\left[-\delta,0\right];\overline{\mathrm{Dom}\left(\varphi\right)}\big) \\ &V(T,\xi)=h(X(0),\bar{X}^{\lambda,\delta}(0)),\ \xi\in C\big(\left[-\delta,0\right];\overline{\mathrm{Dom}\left(\varphi\right)}\big) \end{aligned} \tag{1.7}$$

Our aim is to prove that V satisfies the dynamic programming principle and is a viscosity solution of a proper Hamilton-Jacobi-Bellman equation.
In the second part, we establish necessary conditions for the optimal control u^* minimizing the cost functional

$$J(u):=\mathbb{E}\bigg[\int_0^T g(t,R(X)(t),u(t))dt+h(X(T))\bigg] \tag{1.8}$$

subject to the one-dimensional stochastic variational inequality (SVI) with delay

$$\begin{cases} dX(t)+\partial\varphi(X(t))dt\ni b(t,R(X)(t),u(t))dt \\ \qquad\qquad\qquad +\sigma(t,R(X)(t),u(t))dW(t),\ t\in[0,T]; \\ X(t)=\eta(t),\ t\in[-\delta,0]. \end{cases} \tag{1.9}$$

where $\partial\varphi$ is the subdifferential of a lower semi-continuous (l.s.c.) convex function φ and

$$R(X)(t):=\int_{-\delta}^0 X(t+r)\,d\alpha(r),\ t\in[0,T]$$

is a delay term applied to the dynamics of the system. In order to reach this goal we will employ one of the essential approaches in solving optimal control problems, the maximum principle.

Chapter 2

Preliminaries

2.1 Elements of nonlinear analysis

We begin by recalling definitions and properties of maximal monotone operators. Then we remind bounded variation functions, mentioning that the space of such functions is a Banach space and is the dual of the space of continuous functions. Finally, we come to proper convex lower semicontinuous functions, the subdifferential of such functions are maximal monotone operators.

2.2 Elements of stochastic analysis

We start with the classical stochastic calculus : Brownian motion, stochastic integral and stochastic differential equations. Then we recall some existence and uniqueness results for stochastic delayed differential equations and backward stochastic differential equations with time delayed generators. Finally we bring back to mind equations of multivalued types, by giving existence and uniqueness results for multivalued stochastic differential equations and multivalued backward stochastic differential equations.

Chapter 3

Multivalued stochastic differential equations with delay

In this chapter[1] the equation envisaged is:

$$\begin{cases} dX(t) + \partial\varphi\left(X(t)\right)dt \ni b\left(t, X(t), \Theta(t), \Pi(t)\right)dt \\ \qquad\qquad\qquad\qquad + \sigma\left(t, X(t), \Theta(t), \Pi(t)\right)dW(t),\ t \in (0, T], \\ X(t) = \xi\left(t\right),\ t \in \left[s-\delta, s\right], \end{cases} \tag{3.1}$$

where

$$\Pi(t) := X(t-\delta), \quad \Theta(t) := \int_{-\delta}^{0} e^{\lambda r} X(t+r)dr = e^{-\lambda t}\int_{t-\delta}^{t} e^{\lambda s} X(s)\,ds \tag{3.2}$$

with $\delta \geq 0$ is a fixed delay, $\lambda \in \mathbb{R}$ and $\xi \in C\big(\left[-\delta, 0\right]; \overline{\mathrm{Dom}\left(\varphi\right)}\big)$ is arbitrary fixed.
We recall that the existence problem for stochastic equation (3.1) without the multivalued term $\partial\varphi$ has been treated by Mohammed in (18) (see also (17)). On the other hand, the variational inequality without delay has been considered by Bensoussan & Răşcanu in (3) (for the first time) and Asiminoaei & Răşcanu in (1) (where the existence is proved through a penalized method). After that the results have been extended by Răşcanu in (21) (the Hilbert space framework) and Cépa in (6) (the finite dimensional case) by considering a maximal monotone operator A instead of $\partial\varphi$.

[1]The results of this chapter are part of a joint paper (9) submitted for publication

3.1 Assumptions

Let $s \in [0,T)$ be arbitrary but fixed and $(\Omega, \mathcal{F}, \{\mathcal{F}_t^s\}_{t\geq s}, \mathbb{P})$ be a stochastic basis. The process $\{W(t)\}_{t\geq s}$ is a d-dimensional standard Brownian motion with respect to this basis.

We will need the following assumptions:

(H_1) The function $\varphi : \mathbb{R}^d \to (-\infty, +\infty]$ is convex and lower semicontinuous (l.s.c.) such that

$$\mathrm{Int}\,(\mathrm{Dom}\,(\varphi)) \neq \emptyset,$$

and we suppose that

$$0 \in \mathrm{Int}\,(\mathrm{Dom}\,(\varphi)) \quad \text{and} \quad \varphi(x) \geq \varphi(0) = 0,\ \forall\, x \in \mathbb{R}^d;$$

Remark 3.1.1. *We choose a specific φ, the indicator function of a non-empty closed convex subset $\bar{D}$ of $\mathbb{R}^d$, $I_{\bar{D}} : \mathbb{R}^d \to (-\infty, +\infty]$. In this case, the supplementary drift $\partial I_{\bar{D}}(X(t))$ is an "inward push" that forbids the process $X(t)$ to leave the domain $\bar{D}$ and this drift acts only when $X(t)$ reach the boundary of $\bar{D}$.*

(H_2) The functions $b : [0,T] \times \mathbb{R}^{3d} \to \mathbb{R}^d$ and $\sigma : [0,T] \times \mathbb{R}^{3d} \to \mathbb{R}^{d\times d}$ are continuous and there exist $\ell, \kappa > 0$ such that for all $t \in [0,T]$ and $x, y, z, x', y', z' \in \mathbb{R}^d$,

$$\begin{aligned}
&|b(t,x,y,z) - b(t,x',y',z')| \leq \ell\,(|x-x'| + |y-y'| + |z-z'|)\,,\\
&|\sigma(t,x,y,z) - \sigma(t,x',y',z')| \leq \ell\,(|x-x'| + |y-y'| + |z-z'|)\,,\\
&|b(t,0,0,0)| + |\sigma(t,0,0,0)| \leq \kappa.
\end{aligned} \tag{3.3}$$

(H_3) The initial path ξ is $\mathcal{F}_s^s$-measurable and

$$\xi \in L^2\big(\Omega; C\big([-\delta, 0]\,; \overline{\mathrm{Dom}\,(\varphi)}\,\big)\big) \quad \text{and} \quad \varphi\,(\xi\,(0)) \in L^1\,(\Omega; \mathbb{R})\,. \tag{3.4}$$

Definition 3.1.1. *A pair of progressively measurable continuous stochastic processes*

$(X,K) : \Omega \times [s-\delta,T] \to \mathbb{R}^{2d}$ *is a solution of (3.1) if*

$$\begin{aligned}
(i) \quad & X \in L^2_{\mathbb{F}}\left(\Omega; C\left([s-\delta,T];\mathbb{R}^d\right)\right), \\
(ii) \quad & X(t) \in \overline{\mathrm{Dom}(\varphi)},\ a.e.\ t \in [s-\delta,T],\ \mathbb{P}\text{-}a.s. \\
& \text{and } \varphi(X) \in L^1(\Omega \times [s-\delta,T];\mathbb{R}), \\
(iii) \quad & K \in L^2_{\mathbb{F}}\left(\Omega; C\left([s,T];\mathbb{R}^d\right)\right) \cap L^1\left(\Omega; \mathrm{BV}\left([s,T];\mathbb{R}^d\right)\right) \\
& \text{with } K(s) = 0,\ \mathbb{P}\text{-}a.s., \\
(iv) \quad & X(t) + K(t) = X(s) + \int_s^t b(r, X(r), \Theta(r), \Pi(r))\, dr \\
& \qquad + \int_s^t \sigma(r, X(r), \Theta(r), \Pi(r))\, dW(r),\ \forall t \in (s,T],\ \mathbb{P}\text{-}a.s. \\
(v) \quad & X(t) = \xi(t-s),\ \forall t \in [s-\delta, s] \\
(vi) \quad & \int_t^{\hat{t}} \langle u - X(r), dK(r) \rangle + \int_t^{\hat{t}} \varphi(X(r))dr \le (\hat{t}-t)\varphi(u), \\
& \qquad \forall u \in \mathbb{R}^d,\ \forall\ 0 \le t \le \hat{t} \le T,\ \mathbb{P}\text{-}a.s.
\end{aligned} \tag{3.5}$$

3.2 A priori estimates

In all that follows, C denotes a constant, which may depend only on ℓ, κ, δ and T, which may vary from line to line.

The next result provides some a priori estimates of the solution. Write $\|\xi\|_{[-\delta,0]}$, for $\|\xi\|_{C([-\delta,0];\mathbb{R}^d)}$.

Proposition 3.2.1. *We suppose that assumptions* $(\mathrm{H}_1 - \mathrm{H}_3)$ *are satisfied. Let* (X,K) *be a solution of equation (3.1). Then there exists a constant* $C = C(\ell,\kappa,\delta,T) > 0$ *such that*

$$\mathbb{E} \sup_{r\in[s,T]} |X(r)|^2 \le C\left(1 + \mathbb{E}\,\|\xi\|^2_{[-\delta,0]}\right).$$

In addition

$$\mathbb{E} \sup_{r\in[s,T]} |\Theta(r)|^2 + \mathbb{E}\int_s^t |\Pi(r)|^2\, dr \le C\left(1 + \mathbb{E}\,\|\xi\|^2_{[-\delta,0]}\right).$$

Proposition 3.2.2. *We suppose that assumptions* $(\mathrm{H}_1 - \mathrm{H}_3)$ *are satisfied. If* $\left(X^{s,\xi}, K^{s,\xi}\right)$ *and* $(X^{s',\xi'}, K^{s',\xi'})$ *are the solutions of (3.1) corresponding to the initial data* (s,ξ) *and*

(s', ξ') *respectively, then there exists* $C = C(\ell, \kappa, \delta, T) > 0$ *such that*

$$\begin{aligned} &\mathbb{E}\sup_{r\in[s\wedge s',t]} |X^{s,\xi}(r) - X^{s',\xi'}(r)|^2 + \mathbb{E}\sup_{r\in[s\wedge s',t]} |K^{s,\xi}(r) - K^{s',\xi'}(r)|^2 \\ &\qquad\qquad \leq C\Big[\Gamma_1 + |s-s'|\left(1 + \mathbb{E}\,\|\xi\|^2_{[-\delta,0]} + \mathbb{E}||\xi'||^2_{[-\delta,0]}\right)\Big], \end{aligned} \tag{3.6}$$

where

$$\Gamma_1 := \mathbb{E}||\xi - \xi'||^2_{[-\delta,0]} + \mathbb{E}\int_{s'-\delta}^{s'} \left|\xi'(r-s) - \xi'\left(r-s'\right)\right|^2 dr. \tag{3.7}$$

3.3 Existence and uniqueness

We state now the main result of this section:

Theorem 3.3.1. *Under assumptions* $(\mathrm{H}_1 - \mathrm{H}_3)$ *equation (3.1) has a unique solution. Moreover, there exists a constant* $C = C(\ell, \kappa, \delta, T) > 0$ *such that*

$$\begin{aligned} &\mathbb{E}\sup_{r\in[s,T]} |X(r)|^2 + \mathbb{E}\sup_{r\in[s,T]} |K(r)|^2 + \mathbb{E}\,\|K\|_{\mathrm{BV}\left([-\delta,T];\mathbb{R}^d\right)} + \mathbb{E}\int_s^T \varphi(X(r))\,dr \\ &\qquad\qquad \leq C\left(1 + \mathbb{E}\,\|\xi\|^2_{[-\delta,0]}\right) \end{aligned}$$

and

$$\mathbb{E}\sup_{r\in[s,T]} |X(r)|^4 + \mathbb{E}\,\|K\|^2_{\mathrm{BV}\left([-\delta,T];\mathbb{R}^d\right)} + \mathbb{E}\Big(\int_s^T \varphi(X(r))\,dr\Big)^2 \leq C\left(1 + \mathbb{E}\,\|\xi\|^4_{[-\delta,0]}\right).$$

Remark 3.3.1. *The existence of a solution for (3.1) will be shown using the penalized method. More precisely we consider* φ_ε *the Moreau-Yosida regularization of* φ*. The penalized equation is given by:*

$$\begin{cases} dX_\varepsilon(t) + \nabla\varphi_\varepsilon(X_\varepsilon(t))\,dt = b(t, X_\varepsilon(t), \Theta_\varepsilon(t), \Pi_\varepsilon(t))\,dt \\ \qquad\qquad\qquad + \sigma(t, X_\varepsilon(t), \Theta_\varepsilon(t), \Pi_\varepsilon(t))\,dW(t),\ t \in (0,T], \\ X_\varepsilon(t) = \xi(t),\ t \in [-\delta, 0], \end{cases}$$

where

$$\Theta_\varepsilon(t) := \int_{-\delta}^{0} e^{\lambda r} X_\varepsilon(t+r)dr, \quad \Pi_\varepsilon(t) := X_\varepsilon(t-\delta).$$

Chapter 4

Multivalued BSDEs with time-delayed generator

The aim of this chapter[1] is to prove the existence and uniqueness of a solution $(Y(t), Z(t))_{t\in[0,T]}$ for the following multivalued BSDE with time delay generator (formally written as):

$$\begin{cases} -dY(t) + \partial\varphi(Y(t))\,dt \ni F(t, Y(t), Z(t), Y_t, Z_t)\,dt \\ \qquad\qquad\qquad\qquad\qquad +Z(t)\,dW(t),\ 0 \le t \le T, \\ Y(T) = \xi\ . \end{cases} \tag{4.1}$$

where the generator F at time $t \in [0,T]$ depends on the past values of the solution through Y_t and Z_t defined by

$$Y_t := (Y(t+\theta))_{\theta\in[-T,0]} \quad \text{and} \quad Z_t := (Z(t+\theta))_{\theta\in[-T,0]}\ . \tag{4.2}$$

We mention that we will take $Z(t) = 0$ and $Y(t) = Y(0)$ for any $t < 0$.

Delong and Imkeller were the first who introduced and studied in (7) the BSDE with time-delayed generator by considering the equation

$$Y(t) = \xi + \int_t^T F(s, Y_s, Z_s)ds - \int_t^T Z(s)dW(s),\ 0 \le t \le T. \tag{4.3}$$

The mentioned authors have obtained in (7) the existence and uniqueness of the solution for (4.3) if the time horizon T or the Lipschitz constant for the generator F are sufficiently small. Concerning the multivalued term we precise that BSDE involving a

[1]The results of this chapter are part of a joint paper (8) submitted for publication

subdifferential operator (which are also called backward stochastic variational inequalities, BSVI) has been treated by Pardoux and Răşcanu in (20) where they prove the existence and the uniqueness for the equation

$$Y(t) + \int_t^T K(s)\,ds = \xi + \int_t^T F(s, Y(s), Z(s))ds - \int_t^T Z(s)dW(s), \tag{4.4}$$

where $K(t)$ is an element from $\partial\varphi(Y(t))$, and they generalized the Feymann-Kac type formula in order to represent the solution of a multivalued parabolic partial differential equation (PDE). We should mention that the solution Y is reflected at the boundary of the domain of $\partial\varphi$ and the role of the process K is to push Y in order to keep it in this domain. There is a recent paper (16) where it is studied, in the infinite dimensional framework, a generalized version of (4.4) considered on a random time interval (and their applications to the stochastic PDE).

4.1 Assumptions

Let $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})$ be a stochastic basis, $T \in (0, \infty)$ be a finite time horizon, $\{W(t)\}_{t\in[0,T]}$ be a d'-dimensional standard Brownian motion with respect to the stochastic basis and $\mathbb{F} = \{\mathcal{F}_t^W\}_{t\in[0,T]}$. .

The following assumptions will be needed:

(H$_4$) The function $F : \Omega \times [-T, T] \times \mathbb{R}^d \times \mathbb{R}^{d\times d'} \times C([-T, 0]; \mathbb{R}^d) \times \mathbb{L}^2([-T, 0]; \mathbb{R}^{d\times d'}) \to \mathbb{R}^d$ satisfies that there exist $L, \bar{L} > 0$ such that, for some probability measure α on $\mathcal{B}([-T, 0])$ and for any $t \in [0, T]$, $(y, z), (\bar{y}, \bar{z}) \in \mathbb{R}^d \times \mathbb{R}^{d\times d'}$, $(y_t, z_t), (\bar{y}_t, \bar{z}_t) \in C([-T, 0]; \mathbb{R}^d) \times \mathbb{L}^2([-T, 0]; \mathbb{R}^{d\times d'})$, $\mathbb{P}$-a.s.

(i) $F(\cdot, \cdot, y, z, y_\cdot, z_\cdot)$ is $\mathcal{F}_t$-progressively measurable;

(ii) $|F(t, y, z, y_t, z_t) - F(t, \bar{y}, \bar{z}, y_t, z_t)| \le L(|y - \bar{y}| + |z - \bar{z}|)$;

(iii) $$|F(t, y, z, y_t, z_t) - F(t, y, z, \bar{y}_t, \bar{z}_t)|^2 \le \bar{L}\int_{-T}^0 |y(t+\theta) - \bar{y}(t+\theta)|^2\,\alpha(d\theta) + \bar{L}\int_{-T}^0 |z(t+\theta) - \bar{z}(t+\theta)|^2\,\alpha(d\theta)\ ;$$

and

(iv) $$\mathbb{E}\Big[\int_0^T |F(t, 0, 0, 0, 0)|^2\,dt\Big] < \infty\ ;$$

(v) $F(t, \cdot, \cdot, \cdot, \cdot) = 0$, $\forall t < 0$.

(H_5) The function $\varphi : \mathbb{R}^d \to (-\infty, +\infty]$ is proper convex and l.s.c., and we assume

$$\varphi(y) \geq \varphi(0) = 0, \forall\, y \in \mathbb{R}^d.$$

(H_6) The terminal data $\xi : \Omega \to \mathbb{R}^d$ is a $\mathcal{F}_T$-measurable random variable such that

$$\mathbb{E}[|\xi|^2 + |\varphi(\xi)|] < \infty.$$

Let $\mathbb{H}^2_T(\mathbb{R}^{d\times d'})$ denote the space of p.m.s.p. $Z : \Omega \times [0,T] \to \mathbb{R}^{d\times d'}$ satisfying

$$\mathbb{E}\int_0^T |Z(t)|^2\, dt < \infty.$$

Let $\mathbb{S}^2_T(\mathbb{R}^d)$ denote the space of $\mathbb{F}$-adapted and continuous processes $Y : \Omega \times [0,T] \to \mathbb{R}^d$ satisfying

$$\mathbb{E} \sup_{t\in[0,T]} |Y(t)|^2 < \infty.$$

Definition 4.1.1. *The triple (Y, Z, K) is a solution of time-delayed multivalued BSDE (4.1) if*

$$\begin{aligned}
&(i) \quad (Y,Z,K) \in \mathbb{S}^2_T(\mathbb{R}^d) \times \mathbb{H}^2_T(\mathbb{R}^{d\times d'}) \times \mathbb{H}^2_T(\mathbb{R}^d)\ ,\\
&(ii) \quad \mathbb{E}\Big[\int_0^T \varphi\left(Y\left(t\right)\right) dt\Big] < \infty,\\
&(iii) \quad \left(Y\left(t\right), K(t)\right) \in \partial\varphi,\ \mathbb{P}\left(d\omega\right) \otimes dt,\ a.e.\ on\ \Omega \times [0,T],\\
&(iv) \quad Y(t) + \int_t^T K(s)ds = \xi + \int_t^T F(s, Y(s), Z(s), Y_s, Z_s)ds\\
&\qquad\qquad - \textstyle\int_t^T Z(s)dW(s), \forall t \in [0,T],\ a.s.
\end{aligned} \tag{4.5}$$

Remark 4.1.1. *It is easy to show that if $(Y,Z) \in \mathbb{S}^2_T(\mathbb{R}^d) \times \mathbb{H}^2_T(\mathbb{R}^{d\times d'})$ then the generator is well defined and $\mathbb{P}$-integrable, since the following inequality holds true:*

$$\begin{aligned}
\int_0^T |F(s,Y(s),Z(s),Y_s,Z_s)|^2\, ds \leq 3\left(L^2 + \bar{L}\right) T \sup_{t\in[0,T]} |Y\left(s\right)|^2 + 3\left(L^2+\bar{L}\right)\int_0^T |Z(s)|^2\, ds\\
+3\int_0^T |F(s,0,0,0,0)|^2\, ds.
\end{aligned} \tag{4.6}$$

4.2 Existence and uniqueness

Throughout this section C will be a constant (possibly depending on L) which may vary from line to line.

In order to obtain the uniqueness of the solution we will prove the next a priori estimates.

Proposition 4.2.1. *Let assumptions (H_4–H_6) be satisfied. Let $(Y, Z, K), (\bar{Y}, \bar{Z}, \bar{K}) \in \mathbb{S}_T^2(\mathbb{R}^d) \times \mathbb{H}_T^2(\mathbb{R}^{d\times d'}) \times \mathbb{H}_T^2(\mathbb{R}^d)$ be the solutions of (4.1) corresponding to (ξ, F) and $(\bar{\xi}, \bar{F})$ respectively. If the time horizon T or Lipschitz constant $\bar{L}$ are small enough then there exists some constants $C_1 = C_1(L) > 0$ and $C_2 = C_2(L) > 0$, independent of $\bar{L}$ and T, such that*

$$||Y - \bar{Y}||^2_{\mathbb{S}_T^2(\mathbb{R}^d)} + ||Z - \bar{Z}||^2_{\mathbb{H}_T^2(\mathbb{R}^{d\times d'})} \leq C_1 e^{C_2 T}\mathbb{E}\Big[|\xi - \bar{\xi}|^2 + \mathbb{E}\int_0^T \big|F(s, Y(s), Z(s), Y_s, Z_s) - \bar{F}(s, \bar{Y}(s), \bar{Z}(s), \bar{Y}_s, \bar{Z}_s)\big|^2\, ds\Big].$$

The main result of this section is given by

Theorem 4.2.1. *Let assumptions (H_4–H_6) be satisfied. If time horizon T or Lipschitz constant $\bar{L}$ are small enough, then there exists a unique solution (Y, Z, K) of (4.1).*

Remark 4.2.1. *In order to prove the existence of the solution we consider the approximating BSDE with time delayed generator:*

$$\begin{aligned} &Y^\varepsilon(t) + \int_t^T \nabla\varphi_\varepsilon(Y^\varepsilon(s))\, ds \\ &= \xi + \int_t^T F(s, Y^\varepsilon(s), Z^\varepsilon(s), Y_s^\varepsilon, Z_s^\varepsilon)\, ds - \int_t^T Z^\varepsilon(s)\, dW(s), \\ &\qquad\qquad\qquad 0 \leq t \leq T,\ \mathbb{P}\text{-}a.s. \end{aligned}$$

Chapter 5

Stochastic optimal control

5.1 Dynamic programming principle

The aim of this section[1] is to prove that the value function satisfies the dynamic programming principle and is a viscosity solution of a partial differential equation of Hamilton-Jacobi-Bellman (HJB) type.
Let's recall that stochastic optimal control associated to a system with delay is very difficult to treat, since the space of initial data is infinite dimensional. Nonetheless it happens that choosing a specific structure of the dependence of the past and under certain conditions, the control problem for systems with delay can be reduced to a finite dimensional problem. In the case of φ being zero, we refer to the paper of Larssen (13) where it is shown, under Lipschitz assumptions of the coefficients f and h, that the value function satisfies the dynamic programming principle. This work allowed Larssen & Risebro in (14) to prove, in the frame of delay systems and under some supplementary assumptions on V, that the value function is a viscosity solution for a Hamilton-Jacobi-Bellman equation.

5.1.1 Problem formulation

Let $(s, \xi) \in [0, T) \times C\big([-\delta, 0]\,; \overline{\mathrm{Dom}\,(\varphi)}\big)$ be arbitrary but fixed, $\mathrm{U} \subset \mathbb{R}^d$ be a given compact set of admissible control values and $u : \Omega \times [s, T] \to \mathrm{U}$ be the control process. We define the class $\mathcal{U}\,[s, T]$ of admissible control strategies as the set of five-tuples

[1]The results of this section are part of a joint paper (9) submitted for publication

$(\Omega, \mathcal{F}, \mathbb{P}, W, u)$ such that: $(\Omega, \mathcal{F}, \{\mathcal{F}_t^s\}_{t\geq s}, \mathbb{P})$ is a stochastic basis; $\{W(t)\}_{t\geq s}$ is a d'-dimensional standard Brownian motion with $W(s) = 0$ and $\mathbb{F} = \{\mathcal{F}_t^s\}_{t\geq s}$ is generated by the Brownian motion augmented by the $\mathbb{P}$-null set in $\mathcal{F}$; the control process $u : \Omega \times [s,T] \to \mathrm{U}$ is an $\mathbb{F}$-adapted process satisfying

$$\mathbb{E}\Big[\int_s^T |f(t, X(t), \Theta(t), u(t))|\, dt + |h(X(T), \Theta(T))|\Big] < \infty.$$

We consider the following stochastic controlled system

$$\begin{cases} dX(t) + \partial\varphi(X(t))\, dt \ni b(t, X(t), \Theta(t), \Pi(t), u(t))\, dt \\ \qquad\qquad\qquad + \sigma(t, X(t), \Theta(t), \Pi(t), u(t))\, dW(t), \; t \in (s,T], \\ X(t) = \xi(t-s), \; t \in [s-\delta, s], \end{cases} \tag{5.1}$$

where

$$\Theta(t) := \int_{-\delta}^0 e^{\lambda r} X(t+r) dr, \quad \Pi(t) := X(t-\delta), \tag{5.2}$$

together with the cost functional

$$J(s,\xi;u) = \mathbb{E}\Big[\int_s^T f\Big(t, X^{s,\xi,u}(t), \Theta^{s,\xi,u}(t), u(t)\Big)\, dt + h\Big(X^{s,\xi,u}(T), \Theta^{s,\xi,u}(T)\Big)\Big]. \tag{5.3}$$

We define the associated value function:

$$V(s,\xi) = \inf_{u\in\mathcal{U}[s,T]} J(s,\xi;u), \; (s,\xi) \in [0,T) \times C\big([-\delta,0]; \overline{\mathrm{Dom}(\varphi)}\big). \tag{5.4}$$

As it can be seen in the previous section, the following three assumptions will be needed to ensure the existence of a solution $X^{s,\xi,u}$ for (5.1):

(H_7) The function $\varphi : \mathbb{R}^d \to (-\infty, +\infty]$ is convex and l.s.c. such that $\mathrm{Int}(\mathrm{Dom}(\varphi)) \neq \emptyset$ and $\varphi(x) \geq \varphi(0) = 0, \; \forall\, x \in \mathbb{R}^d$.

(H_8) The initial path ξ is $\mathcal{F}_s^s$-measurable such that

$$\xi \in L^2\big(\Omega; C\big([-\delta,0]; \overline{\mathrm{Dom}(\varphi)}\big)\big), \quad \text{and} \quad \varphi(\xi(0)) \in \mathbb{L}^1(\Omega; \mathbb{R}^d).$$

(H_9) The functions $b : [0,T] \times \mathbb{R}^{3d} \times \mathrm{U} \to \mathbb{R}^d$ and $\sigma : [0,T] \times \mathbb{R}^{3d} \times \mathrm{U} \to \mathbb{R}^{d\times d'}$ are continuous and there exist $\ell, \kappa > 0$ such that for all $t \in [0,T]$, $u \in U$ and $x, y, z, x', y', z' \in \mathbb{R}^d$,

$$\begin{aligned} &|b(t,x,y,z,u) - b(t,x',y',z',u)| + |\sigma(t,x,y,z,u) - \sigma(t,x',y',z',u)| \\ &\qquad\qquad \leq \ell\left(|x-x'| + |y-y'| + |z-z'|\right), \end{aligned} \tag{5.5}$$

$$|b(t,0,0,0,u)| + |\sigma(t,0,0,0,u)| \leq \kappa.$$

Theorem 5.1.1. *Under assumptions* $(\mathrm{H}_7\text{–}\mathrm{H}_9)$, *for any* $(s,\xi) \in [0,T)\times C\big([-\delta,0]\,;\overline{\mathrm{Dom}\,(\varphi)}\,\big)$ *and* $u \in \mathcal{U}\,[s,T]$ *there exists a unique pair of processes* $(X,K) = \big(X^{s,\xi,u}, K^{s,\xi,u}\big)$ *which is the solution of the stochastic variational inequality with delay (5.1). In addition, for any* $q \geq 1$, *there exist some constants* $C = C\,(\ell,\kappa,\gamma,T,q) > 0$ *and* $C' = C'\,(\ell,\kappa,\gamma,T,q) > 0$ *such that, for any* (s,ξ), $(s',\xi') \in [0,T) \times C\big([-\delta,0]\,;\overline{\mathrm{Dom}\,(\varphi)}\,\big)$,

$$\begin{aligned}\mathbb{E}\sup_{r\in[s,T]}\Big|X^{s,\xi,u}\,(r)\Big|^{2q} + \mathbb{E}\sup_{r\in[s,T]}\Big|K^{s,\xi,u}\,(r)\Big|^{2q} + \mathbb{E}\,\Big\|K^{s,\xi,u}\Big\|^{q}_{\mathrm{BV}([-\delta,T])}\\ +\mathbb{E}\Big(\int_s^T \varphi\,\Big(X^{s,\xi,u}\,(r)\Big)\,dr\Big)^q \leq C\big[1+\|\xi\|^{2q}_{[-\delta,0]}\,\big]\end{aligned} \tag{5.6}$$

and

$$\begin{aligned}\mathbb{E}\sup\nolimits_{r\in[s\wedge s',t]}|X^{s,\xi}\,(r) - X^{s',\xi'}\,(r)\,|^2 + \mathbb{E}\sup\nolimits_{r\in[s\wedge s',t]}|K^{s,\xi}\,(r) - K^{s',\xi'}\,(r)\,|^2\\ \leq C'\Big[\Gamma_1 + |s-s'|\,\big(1+\|\xi\|^2_{[-\delta,0]} + ||\xi'||^2_{[-\delta,0]}\big)\Big],\end{aligned} \tag{5.7}$$

where

$$\Gamma_1 = ||\xi-\xi'||^2_{[-\delta,0]} + \int_{s'-\delta}^{s'}\big|\xi'\,(r-s) - \xi'\,\big(r-s'\big)\big|^2\,dr \tag{5.8}$$

(as in (3.7)).

Remark 5.1.1. *Using the above estimations and definition (5.2), it is easy to deduce that*

$$\mathbb{E}\sup_{r\in[s,T]}|\Theta^{s,\xi,u}\,(r)\,|^{2q} \leq C\big[1+\|\xi\|^{2q}_{[-\delta,0]}\,\big] \tag{5.9}$$

and

$$\mathbb{E}\sup_{r\in[s\wedge s',t]}|\Theta^{s,\xi}\,(r) - \Theta^{s',\xi'}\,(r)\,|^2 \leq C'\Big[\Gamma_1 + \big|s-s'\big|\,\big(1+\|\xi\|^2_{[-\delta,0]} + ||\xi'||^2_{[-\delta,0]}\big)\Big]. \tag{5.10}$$

5.1.2 Properties of the value function

Under the next assumption the cost functional and the value function will be well-defined.

(H_{10}) The functions $f : [0,T]\times\mathbb{R}^{2d}\times \mathrm{U} \to \mathbb{R}$, $h : \mathbb{R}^{2d} \to \mathbb{R}$ are continuous and there exists $\bar\kappa > 0$ and $p \geq 1$ such that for all $t \in [0,T]$, $u \in U$ and $x, y \in \mathbb{R}^d$,

$$|f\,(t,x,y,u)| + |h\,(x,y)| \leq \bar\kappa\,(1+|x|^p+|y|^p)\,.$$

In the sequel we will follow the techniques from (22) in order to give some basic properties of the value function (including the continuity).

Proposition 5.1.1. *Let assumptions* $(\mathrm{H}_7 - \mathrm{H}_{10})$ *be satisfied. Then there exist* $C > 0$ *such that*

$$|V(s,\xi)| \leq C\ \left[1 + \|\xi\|^p_{[-\delta,0]}\right],\ \forall\,(s,\xi) \in [0,T] \times C\big([-\delta,0]\,;\overline{\mathrm{Dom}\,(\varphi)}\big) \tag{5.11}$$

and

$$\begin{aligned} |V(s,\xi) - V(s',\xi')| \leq\ & C\,\mu_{f,h}(\gamma,M) + C\big[1 + \|\xi\|^p_{[-\delta,0]} + \|\xi'\|^p_{[-\delta,0]}\big]\cdot \\ & \Big[\frac{\Gamma_1^{1/2} + |s-s'|^{1/2}\left(1 + \|\xi\|_{[-\delta,0]} + \|\xi'\|_{[-\delta,0]}\right)}{\gamma} \\ & + \frac{1 + \|\xi\|_{[-\delta,0]} + \|\xi'\|_{[-\delta,0]}}{M}\Big], \\ & \forall\,(s,\xi)\,,(s',\xi') \in [0,T] \times C\big([-\delta,0]\,;\overline{\mathrm{Dom}\,(\varphi)}\big)\,, \end{aligned} \tag{5.12}$$

where $\mu_{f,h}(\gamma,M)$ *is the modulus of continuity of* f *and* h,

$$\mu_{f,h}(\gamma,M) := \sup_{\substack{|x|+|x'|+|y|+|y'|\leq M \\ |x-x'|+|y-y'|\leq\gamma \\ (t,u)\in[0,T]\times \mathrm{U}}} \big\{|f(t,x,y,u) - f(t,x',y',u)| + |h(x,y) - h(x',y')|\big\}\,,$$

for $\gamma, M > 0$.

Definition 5.1.1. *We say that the value function satisfies the dynamic programming principle (DPP for short) if, for every* $(s,\xi) \in [0,T) \times C\big([-\delta,0]\,;\ \overline{\mathrm{Dom}\,(\varphi)}\big)$, *it holds that*

$$V(s,\xi) = \inf_{u\in\mathcal{U}[s,T]} \mathbb{E}\Big[\int_s^\theta f\left(t, X^{s,\xi,u}(t), \Theta^{s,\xi,u}(t), u(t)\right) dt + V\left(\theta, X^{s,\xi,u}(\theta)\right)\Big], \tag{5.13}$$

for every stopping time $\theta \in [s,T]$.

In order to show that V satisfies the DPP, we consider, for $\epsilon > 0$, the penalized equation:

$$\begin{cases} dX_\epsilon(t) + \nabla\varphi_\epsilon(X_\epsilon(t))\,dt = b\,(t, X_\epsilon(t)\,,\Theta_\epsilon(t)\,,\Pi_\epsilon(t)\,,u(t))\,dt \\ \qquad\qquad +\sigma\,(t, X_\epsilon(t)\,,\Theta_\epsilon(t)\,,Z_\epsilon(t)\,,u(t))\,dW(t),\ t \in (s,T], \\ X_\epsilon(t) = \xi\,(t-s)\,,\ t \in [s-\delta,s]\,, \end{cases} \tag{5.14}$$

where

$$\Theta_\epsilon(t) := \int_{-\delta}^0 e^{\lambda r} X_\epsilon(t+r)dr, \quad \Pi_\epsilon(t) := X_\epsilon(t-\delta) \tag{5.15}$$

and we take the associated penalized value function

$$V_\epsilon(s,\xi) = \inf_{u\in\mathcal{U}[s,T]} \mathbb{E}\Big[\int_s^T f(t, X_\epsilon^{s,\xi,u}(t), \Theta_\epsilon^{s,\xi,u}(t), u(t))\,dt + h(X_\epsilon^{s,\xi,u}(T), \Theta_\epsilon^{s,\xi,u}(T))\Big],$$
$$(s,\xi) \in [0,T) \times C\big([-\delta,0];\overline{\mathrm{Dom}(\varphi)}\big). \tag{5.16}$$

Remark 5.1.2. *Inequalities (5.11) and (5.12) hold true for the penalized value function* V_ϵ.

The following result is a straightforward generalization of Theorem 4.2 from (13) to the case where f and h satisfy only sublinear growth and continuity (instead of lipschitz continuity):

Lemma 5.1.1. *Let assumptions* $(\mathrm{H}_7 - \mathrm{H}_{10})$ *be satisfied. If* $X_\epsilon^{s,\xi,u}$ *is the solution of (5.14), then, for every* $(s,\xi) \in [0,T) \times C\big([-\delta,0];\overline{\mathrm{Dom}(\varphi)}\big)$*, it holds that*

$$V_\epsilon(s,\xi) = \inf_{u\in\mathcal{U}[s,T]} \mathbb{E}\Big[\int_s^\tau f(r, X_\epsilon^{s,\xi,u}(r), \Theta_\epsilon^{s,\xi,u}(r), u(r))dr + V_\epsilon(\tau, X_\epsilon^{s,\xi,u}(\tau))\Big], \tag{5.17}$$

for every stopping time $\tau \in [s,T]$.

Proof. Since the equation (5.14) has Lipschitz coefficients, we can use the proof of Theorem 4.2 in (13) also replacing the inequality from Lemma 4.1 by (5.12). $\square$

Proposition 5.1.2. *Let assumptions* $(\mathrm{H}_7 - \mathrm{H}_{10})$ *be satisfied. Then there exists* $C > 0$ *such that*

$$\begin{aligned} |V_\epsilon(s,\xi) - V(s,\xi)| \le{}& C\,\mu_{f,h}(\gamma, M) + C\big(1 + \|\xi\|_{[-\delta,0]}^p\big)\cdot \\ & (1 + \varphi^{1/4}(\xi(0)) + \|\xi\|_{[-\delta,0]})\left[\frac{\epsilon^{1/16}}{\gamma} + \frac{1}{M}\right], \\ & \forall\,(s,\xi) \in [0,T] \times C\big([-\delta,0];\overline{\mathrm{Dom}(\varphi)}\big). \end{aligned} \tag{5.18}$$

Using, mainly inequalities (5.12) and (5.18) we can prove that

Lemma 5.1.2. *Function* V_ϵ *is uniformly convergent on compacts to the value function* V *on* $[0,T] \times C\big([-\delta,0];\overline{\mathrm{Dom}(\varphi)}\big)$.

The main result of this section is the following:

Proposition 5.1.3. *Under the assumptions* $(\mathrm{H}_7 - \mathrm{H}_{10})$ *the value function* V *satisfies the DPP.*

5.1.3 Hamilton-Jacobi-Bellman Equation. Viscosity solution

Since V is defined on $[0,T] \times C\big([-\delta,0];\overline{\mathrm{Dom}(\varphi)}\big)$, the associated Hamilton-Jacobi-Bellman equation will be an infinite dimensional PDE. In general the value function $V(s,\xi)$ depend on the initial path in a complicated way. In order to simplify the problem, our conjecture will be that the value function V depends on ξ only through (x,y) where

$$x = x(\xi) := \xi(0) \quad \text{and} \quad y = y(\xi) := \int_{-\delta}^{0} e^{\lambda r}\xi(r)\,dr.$$

Hence the problem can be reduced to a finite dimensional optimal control problem by working with a new value function $\tilde{V}$ given by

$$\tilde{V} : [0,T] \times \mathbb{R}^{2d} \to \mathbb{R}, \quad \tilde{V}(s,x,y) := V(s,\xi).$$

Our aim is to prove that the value function $\tilde{V}$ is a viscosity solution of the following Hamilton-Jacobi-Bellman type PDE

$$\begin{cases} -\dfrac{\partial \tilde{V}}{\partial s}(s,x,y) + \sup\limits_{u\in \mathrm{U}} \mathcal{H}\big(s,x,y,z,u,-D_x\tilde{V}(s,x,y), -D^2_{xx}\tilde{V}(s,x,y)\big) \\ \qquad -\langle x - e^{-\lambda\delta}z - \lambda y, D_y\tilde{V}(s,x,y)\rangle \in \langle -D_x\tilde{V}(s,x,y), \partial\varphi(x)\rangle, \\ \qquad\qquad \text{for } (s,x,y,z) \in (0,T) \times \overline{\mathrm{Dom}(\varphi)} \times \mathbb{R}^{2d}, \\ \tilde{V}(T,x,y) = h(x,y) \ \text{ for } (x,y) \in \overline{\mathrm{Dom}(\varphi)} \times \mathbb{R}^d, \end{cases} \tag{5.19}$$

where $\mathcal{H} : [0,T] \times \mathbb{R}^{3d} \times \mathrm{U} \times \mathbb{R}^d \times \mathbb{R}^{d\times d} \to \mathbb{R}$ is defined by

$$\mathcal{H}(s,x,y,z,u,q,p) := \langle b(s,x,y,z,u), q\rangle + \frac{1}{2}\mathrm{Tr}(\sigma\sigma^*)(s,x,y,z,u)\,p - f(s,x,y,u).$$

Let us define, for $x \in \overline{\mathrm{Dom}(\varphi)}$ and $z \in \mathbb{R}^d$,

$$\partial\varphi_*(x;z) = \liminf_{\substack{(x',z')\to(x,z)\\ x^*\in\partial\varphi(x')}} \big\langle x^*, z'\big\rangle \quad \text{and} \quad \partial\varphi^*(x;z) = \limsup_{\substack{(x',z')\to(x,z)\\ x^*\in\partial\varphi(x')}} \big\langle x^*, z'\big\rangle. \tag{5.20}$$

Remark 5.1.3. *Obviously,* $\partial\varphi^*(x;z) = -\partial\varphi_*(x;-z)$.

The following technical result can be found in (22):

Lemma 5.1.3. (i) *For any* $x \in \mathrm{Int}(\mathrm{Dom}(\varphi))$ *and* $z \in \mathbb{R}^d$

$$\partial\varphi_*(x;z) = \inf_{x^*\in\partial\varphi(x)} \langle x^*, z\rangle \tag{5.21}$$

(ii) For any $x \in \mathrm{Bd}\,(\mathrm{Dom}\,(\varphi))$ *and* $z \in \mathbb{R}^d$ *such that*

$$\inf_{\mathrm{n} \in \mathcal{N}(x)} \langle \mathrm{n}, z \rangle > 0$$

equality (5.21) still holds

(here $\mathcal{N}(x)$ *denotes the exterior normal cone, in a point* x *which belongs to the boundary of the domain).*

It is easy to see that in the particular case of φ being the indicator function of a closed convex set $\mathcal{K}$ (i.e. $\varphi(x) = 0$, if $x \in \mathcal{K}$ and $\varphi(x) = +\infty$ if $x \notin \mathcal{K}$), we obtain the form:

$$\partial\varphi_*(x; z) = \begin{cases} 0, & \text{if } x \in \mathrm{Int}\,(\mathrm{Dom}\,(\varphi)) \text{ or if } x \in \mathrm{Bd}\,(\mathrm{Dom}\,(\varphi)) \text{ with } \inf_{\mathrm{n} \in \mathcal{N}(x)} \langle \mathrm{n}, z \rangle > 0, \\ -\infty, & \text{if } \inf_{\mathrm{n} \in \mathcal{N}(x)} \langle \mathrm{n}, z \rangle \leq 0. \end{cases}$$

We define the viscosity solution for HJB equation (5.19):

Definition 5.1.2. *Let* $v : (0, T] \times \overline{\mathrm{Dom}\,(\varphi)} \times \mathbb{R}^d \to \mathbb{R}$ *be a continuous function which satisfies* $v(T, x, y) = h(x)$, $\forall\ (x, y) \in \overline{\mathrm{Dom}\,(\varphi)} \times \mathbb{R}^d$.

(a) We say that v *is a viscosity subsolution of (5.19) if in any point* $(s, x, y) \in (0, T] \times \overline{\mathrm{Dom}\,(\varphi)} \times \mathbb{R}^d$ *which is a maximum point for* $v - \Psi$, *for any* $\Psi \in C^{1,2,1}((0, T) \times \overline{\mathrm{Dom}\,(\varphi)} \times \mathbb{R}^d; \mathbb{R})$, *the following inequality is satisfied:*

$$\begin{aligned} -\frac{\partial \Psi}{\partial t}(s, x, y) &+ \sup\nolimits_{u \in \mathrm{U}} \mathcal{H}\left(s, x, y, z, u, -D_x\Psi(s, x, y), -D_{xx}^2\Psi(s, x, y)\right) \\ &- \left\langle x - e^{-\lambda\delta} z - \lambda y, D_y\Psi(s, x, y)\right\rangle \leq \partial\varphi^*(x; -D_x\Psi(t, x, y)). \end{aligned}$$

(b) We say that v *is a viscosity supersolution of (5.19) if in any point* $(s, x, y) \in (0, T] \times \overline{\mathrm{Dom}\,(\varphi)} \times \mathbb{R}^d$ *which is a minimum point for* $v - \Psi$, *for any* $\Psi \in C^{1,2,1}((0, T) \times \overline{\mathrm{Dom}\,(\varphi)} \times \mathbb{R}^d; \mathbb{R})$, *the following inequality is satisfied:*

$$\begin{aligned} -\frac{\partial \Psi}{\partial t}(s, x, y) &+ \sup\nolimits_{u \in \mathrm{U}} \mathcal{H}\left(s, x, y, z, u, -D_x\Psi(s, x, y), -D_{xx}\Psi(s, x, y)\right) \\ &- \left\langle x - e^{-\lambda\delta} z - \lambda y, D_y\Psi(s, x, y)\right\rangle \geq \partial\varphi_*(x; -D_x\Psi(t, x, y)). \end{aligned}$$

(c) We say that v *is a viscosity solution of (5.19) if it is both a viscosity sub- and super-solution.*

Theorem 5.1.2. *Under assumptions* $(\mathrm{H}_7 - \mathrm{H}_{10})$ *the value function* $\tilde{V}$ *is a viscosity solution of (5.19).*

5.2 Necessary conditions of optimality

In this section[1] we will use the maximum principle approach. It has been introduced by Pontryagin and his group in the 1950's to establish necessary conditions of optimality for deterministic controlled systems. Since then, the number of papers on the subject sharply increased and a lot of work has been done on different type of systems. One major difficulty that arises in the extension to the stochastic controlled systems is that the adjoint equation becomes an SDE with terminal conditions, called backward SDEs (BSDEs). Pioneering work in this direction was achieved by Kushner (12), Bismut (4) or Haussman (11). Concerning the control of stochastic delayed differential equations, Oksendal & Sulem & Zhang in (19) established sufficient and necessary stochastic maximum principle, where the associated adjoint equation is a time-advanced backward stochastic differential equation. In the other hand, Barbu (2) initiated systematic studies on controlled variational inequalities in the deterministic case. On stochastic control of stochastic variational inequality, results have been obtained in the following directions: the study of associated Hamilton-Jacobi-Bellman equation ((9), (22)) and necessary conditions of optimality (23).

5.2.1 Statement of the problem

We fix the delay constant $\delta > 0$, the time horizon $T > 0$ and a vector of d finite positive finite scalar measures on $\mathcal{B}([-\delta, 0])$, $\alpha = (\alpha^1, \dots, \alpha^d)$. The space of controls is a convex closed set $U \subseteq \mathbb{R}^l$. For the sake of simplicity, we will suppose that U is also bounded; this is not really a restriction, since usually one needs compactness in order to obtain the existence of an optimal control.

For $-\delta \le s \le t \le T$, we denote by $D[s,t]$ the space of càdlàg functions on $[s,t]$; the sup-norm on $D[s,t]$ is denoted $\|\cdot\|_{s,t}$ (or $\|\cdot\|_t$, if $s = -\delta$).

Let $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})$ be a stochastic basis, W a one-dimensional standard Brownian motion and $\mathbb{F} := \{\mathcal{F}_t^W\}_{t\ge 0}$ the filtration generated by W augmented by the null-sets of $\mathcal{F}$. We prolong the filtration $\{\mathcal{F}_t\}_{t\ge 0}$ on $[-\delta, 0)$ by setting $\mathcal{F}_t := \mathcal{F}_0$ for $t \in [-\delta, 0)$ (and we still denote by $\mathbb{F}$ the filtration $\{\mathcal{F}_t\}_{t\ge -\delta}$).

Sometimes it is interesting to restrict the information available to the controller and consider a subfiltration of $\mathbb{F}$, $\mathbb{G} := \{\mathcal{G}_t\}_{t\ge 0}$, instead of $\mathbb{F}$.

[1]The results of this section are part of a joint paper (10) submitted for publication

We introduce the following spaces:

- $L^p_{\mathbb{F}}(\Omega; C[-\delta, T]))$, $1 \le p \le +\infty$, the linear space of continuous, p-integrable processes on $[-\delta, T]$ which are $\mathbb{F}$-adapted;
- $L^p_{\mathbb{F}}(\Omega; BV[-\delta, T]))$, $1 \le p \le +\infty$, the linear space of bounded variation, p-integrable processes on $[-\delta, T]$ which are $\mathbb{F}$-adapted;
- $L^2_{\mathbb{G}}(\Omega \times [0, T]; U)$, the linear space of square integrable U-valued processes on $[0, T]$ which are progressively measurable with respect to $\mathbb{G}$.

We consider the following SVI with delay

$$\begin{cases} dX(t) + \partial\varphi(X(t))dt \ni b(t, R(X)(t), u(t))dt \\ \qquad\qquad\qquad\qquad + \sigma(t, R(X)(t), u(t))dW(t), \ t \in [0, T], \\ X(t) = \xi(t), \ t \in [-\delta, 0], \end{cases} \tag{5.22}$$

where:

- u is an *admissible control*, *i.e.* u is an U-valued, progressively measurable process with respect to $\mathbb{G}$;
- R is the delay term defined by $R(x)(t) := \int_{-\delta}^{0} x(t + r)d\alpha(r)$ for $x \in C[-\delta, T]$ and $t \in [0, T]$;
- the measurable functions $b : [0, T] \times \mathbb{R}^d \times U \to \mathbb{R}$, $\sigma : [0, T] \times \mathbb{R}^d \times U \to \mathbb{R}$ are the coefficients of the equation;
- $\varphi : \mathbb{R} \to (-\infty, +\infty]$ is a l.s.c. convex function with $\operatorname{int} \operatorname{Dom} \varphi \neq \emptyset$;
- ξ represents the starting deterministic process, satisfying the following condition:

(H_{11}) $\xi \in C[-\delta, 0]$ and $\xi(0) \in \overline{\operatorname{Dom} \varphi}$.

We mention that coefficients depending also on the present state of the solution $X(t)$ can be envisaged by replacing α with $\alpha' := (\alpha, \delta_0)$, where δ_0 is the Dirac measure on $[-\delta, 0]$ concentrated in 0.

Definition 5.2.1. *A pair of continuous $\mathbb{F}$-adapted processes (X, K) is called a solution of (5.22) if the following hold $\mathbb{P}$-a.s.:*

(*i*) $||K||_{BV[0,T]} < \infty$; $K(t) = 0$, $\forall t \in [-\delta, 0]$;

(*ii*) $X(t) = \xi(t)$, $\forall t \in [-\delta, 0]$;

(*iii*) $X(t)+K(t) = \xi(0)+\int_0^t b(s, R(X)(s), u(s))ds+\int_0^t \sigma(s, R(X)(s), u(s))dW(s)$, $\forall\, t \in [0, T]$;

(*iv*) $\int_0^T (y(r) - X(r))dK(r) + \int_0^T \varphi(X(r))dr \leq \int_0^T \varphi(y(r))dr$, $\forall y \in C[0, T]$.

In order to have existence and uniqueness of strong solutions for equation (5.22), we impose the following conditions on the coefficients:

(H_{12}) there exists a constant $L > 0$ such that for every $t \in [0, T]$, $x, \tilde{x} \in C([-\delta, T])$ and $u \in U$:

(*i*) $|b(t, R(x)(t), u) - b(t, R(\tilde{x})(t), \tilde{u})| \leq L||x - \tilde{x}||_t$;

(*ii*) $|\sigma(t, R(x)(t), u) - \sigma(t, R(\tilde{x})(t), \tilde{u})| \leq L||x - \tilde{x}||_t$.

Theorem 5.2.1. *Let $p > 1$. Under assumptions (H_{11}) and (H_{12}), for each control u, equation (5.22) has a unique solution $(X^u, K^u) \in L^p_{\mathbb{F}}(\Omega; C[-\delta, T]))\times\big(L^p_{\mathbb{F}}(\Omega; C[-\delta, T]))\cap L^{p/2}_{\mathbb{F}}(\Omega; BV[-\delta, T]))\big)$.*

Proof. The proof follows the same steps as that of Theorem 3.3.1. □

In order to have continuous dependence on controls, we impose the supplementary Lipschitz condition:

(H_{13}) there exists a constant $\tilde{L} > 0$ such that for every $t \in [0, T]$, $x \in C([-\delta, T])$ and $u, v \in U$:

(*i*) $|b(t, R(x)(t), u) - b(t, R(x)(t), v)| \leq \tilde{L}\,|u - v|$;

(*ii*) $|\sigma(t, R(x)(t), u) - \sigma(t, R(x)(t), v)| \leq \tilde{L}\,|u - v|$.

Proposition 5.2.1. *Under assumptions (H_{11})-(H_{13}), there exists a constant $C > 0$ such that*

$$\mathbb{E}\,\|X^u - X^v\|_T^2 + \mathbb{E}\,\|K^u - K^v\|_T^2 + \mathbb{E}\,\|K^u - K^v\|_{BV[0,T]} \leq C\mathbb{E}\int_0^T |u(t) - v(t)|^2\, dt,$$

for all controls u, $v \in L^2_{\mathbb{G}}(\Omega \times [0, T]\,; U)$.

The purpose of this section is to give necessary conditions of optimality under the form of a maximum principle for the optimal control.

From now on we assume that $\operatorname{Dom}\varphi = \mathbb{R}$. Let us define the second order derivative of φ as the unique σ-finite positive measure μ on $\mathcal{B}(\mathbb{R})$ such that

$$\mu([a,a']) = \varphi'_+(a') - \varphi'_-(a), \text{ if } a \le a',$$

where $\varphi'_-(x)$ and $\varphi'_+(x)$ are the left-hand side, respectively the right-hand side derivatives of φ in x.

(H_{14}) b, g, σ and h are C^1 in $(y,u) \in \mathbb{R}^d \times U$ with uniformly bounded derivatives.

By Theorem 5.2.1, for every control u we have, under conditions (H_{11}) and (H_{14}), the existence of a unique solution $(X^u(t), K^u(t))_{t\in[0,T]} \in L^2_{\mathbb{F}}(\Omega; C[0,T]) \times (L^2_{\mathbb{F}}(\Omega; C[0,T]) \cap L^1_{\mathbb{F}}(\Omega; BV[0,T]))$ for equation (5.22).

(H_{15}) $\sigma(t,y,u) \neq 0, \ \forall (t,y,u) \in [0,T] \times \mathbb{R}^d \times U$.

From now on, (H_{11}), (H_{14}) and (H_{15}) are the standing assumptions.

For an admissible control u we introduce the *local time* of the process X^u by

$$L^{a,u}(t) := |X^u(t) - a| - |X^u(0) - a| - \int_0^t \operatorname{sgn}(X^u(s) - a) dX^u(s).$$

We always can (and will) choose a version which is measurable in $(a,t,\omega) \in \mathbb{R} \times [0,T] \times \Omega$, continuous and increasing in $t \ge 0$, càdlàg in $a \in \mathbb{R}$. We recall here some properties of the local time:

Proposition 5.2.2. *Let u be an admissible control. Then:*

1. *for every bounded, Borel function γ,*

$$\int_{\mathbb{R}} L^{a,u}(t)\gamma(a)da = \int_0^t \gamma(X^u(s)) \left|\sigma(s, R(X^u)(s), u(s))\right|^2 ds, \ t \in [0,T]\,;$$

2. *for every $t \in [0,T]$ and $a \in \mathbb{R}$,*

$$(X^u(t) - a)^+ - (X^u(0) - a)^+ = \int_0^t \mathbf{1}_{\{X^u(s) > a\}} dX^u(s) + \frac{1}{2} L^{a,u}(t);$$

3. for every $t \in [0, T]$ and $a \in \mathbb{R}$,

$$L^{a,u}(t) - L^{a-,u}(t) = 2 \int_0^t \mathbf{1}_{\{X^u(s)=a\}} \left[b(s, R(X^u)(s), u(s)) ds - dK^u(s) \right].$$

Formulas 1. and 2. are called *occupation time density formula*, respectively *Tanaka formula.*

A consequence of (H_{15}) is the absolute continuity of the bounded variation process K^u. Indeed, the formula of time occupation density gives us

$$\int_0^T \mathbf{1}_{\{X^u(t)=x\}} |\sigma(t, R(X^u)(t), u(t))|^2 dt = \int_{\mathbb{R}} \mathbf{1}_{\{a=x\}} L^{a,u}(t) da = 0 \text{ a.s.},$$

which yields

$$\int_0^T \mathbf{1}_{\{X^u(t)=x\}} dt = 0 \text{ a.s., } \forall x \in \overline{\operatorname{Dom} \varphi}.$$

Let us set $\Lambda := \left\{ x \in \overline{\operatorname{Dom} \varphi} \mid \varphi'_+(x) > \varphi'_-(x) \right\}$. Since Λ is at most countable, we obtain $\int_0^T \mathbf{1}_{\{X^u(t) \in \Lambda\}} dt = 0$ a.s. By the inequalities

$$\int_0^t \varphi'_-(X^u(s)) ds \leq K^u(t) \leq \int_0^t \varphi'_+(X^u(s)) ds \text{ a.s.},$$

we obtain equality in this relation; therefore K^u is absolutely continuous. From Proposition 5.2.2-3., this property implies the continuity of $L^{\cdot,u}(t)$.

Now, if we had φ of class C^2, then by occupation time density formula we would have had

$$\begin{aligned} \int_0^t \varphi''(X^u(s)) ds &= \int_{\mathbb{R}} \int_0^t \frac{L^{a,u}(ds)}{|\sigma(s, R(X^u)(s), u(s))|^2} \varphi''(a) da \\ &= \int_{\mathbb{R}} \int_0^t \frac{L^{a,u}(ds)}{|\sigma(s, R(X^u)(s), u(s))|^2} \mu(da). \end{aligned}$$

This serves as a motivation for introducing the increasing process:

$$A^u(t) := \int_{\mathbb{R}} \int_0^t \frac{L^{a,u}(ds)}{|\sigma(s, R(X^u)(s), u(s))|^2} \mu(da), \; t \in [0, T].$$

Since $L^{a,u}(t) = 0$ if $\|X^u\|_t \leq |a|$, it follows that A^u is also finite and continuous.

5.2.2 Variation equation

Let φ_ε be the Moreau-Yosida regularization of φ. We recall that it is a C^1-function approximating φ. By a mollification procedure, we introduce the increasing C^∞-functions $\beta_\varepsilon : \mathbb{R} \to \mathbb{R}$, $\varepsilon > 0$, such that β'_ε, β''_ε are bounded and

$$\left|\beta_\varepsilon(x) - \varphi'_\varepsilon(x)\right| < \varepsilon,\ \forall x \in \mathbb{R}. \tag{5.23}$$

Moreover, if φ is affine outside a compact interval, then $(\beta_\varepsilon)_{\varepsilon\in(0,1]}$ is uniformly bounded and there exists another compact interval I such that $\beta'_\varepsilon(x) \le \varepsilon$, $x \in I^{\mathrm{c}}$.

For the moment, let us fix two controls u^0 and u^1; let us set, for $\theta \in (0,1)$,

$$u^\theta(t) := u^0(t) + \theta(u^1(t) - u^0(t)),\ t \in [0,T].$$

In order to simplify the notations, we write X^θ, K^θ, $L^{a,\theta}$, A^θ instead X^{u^θ}, K^{u^θ}, L^{a,u^θ}, respectively A^{u^θ}. The reason for studying the behavior of X^θ as $\theta \to 0$ is that $\theta \mapsto J(u^\theta)$ has a minimum in $\theta = 0$ if u^0 is an optimal control, hence we can derive necessary optimality conditions by calculating its derivative in 0.

Let $X^{\varepsilon,\theta}$ be the solution of the penalized equation

$$\begin{aligned} dX^{\varepsilon,\theta}(t) + \beta_\varepsilon(X^{\varepsilon,\theta}(t))dt =& b(t, R(X^{\varepsilon,\theta})(t), u^\theta(t))dt \\ &+ \sigma(t, R(X^{\varepsilon,\theta})(t), u^\theta(t))dW(t),\ t \in [0,T]; \end{aligned} \tag{5.24}$$

with initial condition $X^{\varepsilon,\theta}(t) = \xi(t)$ on $[-\delta, 0]$. We set $K^{\varepsilon,\theta}(t) := \int_0^t \beta_\varepsilon(X^{\varepsilon,\theta}(s))ds$, $t \in [0,T]$; $K^{\varepsilon,\theta}(t) := 0,\ t \in [-\delta, 0)$.

From the proof of (1, Theorem 2.1.) and relation (5.23), $X^{\varepsilon,\theta}$ and $K^{\varepsilon,\theta}$ converge as $\varepsilon \searrow 0$ in $L^2_{\mathbb{F}}(\Omega; C[0,T])$ to X^θ, respectively K^θ, uniformly with respect to θ.

We also consider, for $\varepsilon > 0$ and $\theta \in [0,1]$, the solution $Y^{\varepsilon,\theta}$ of the delay equation

$$\begin{aligned} dY^{\varepsilon,\theta}(t) + \beta'_\varepsilon(X^{\varepsilon,\theta}(t))Y^{\varepsilon,\theta}(t)dt = \Big[\langle \partial_y b^{\varepsilon,\theta}(t), R(Y^{\varepsilon,\theta})(t)\rangle + \langle \partial_u b^{\varepsilon,\theta}(t), u^1(t) - u^0(t)\rangle\Big] dt \\ + \Big[(\partial_y \sigma^{\varepsilon,\theta}(t))R(Y^{\varepsilon,\theta})(t) + (\partial_u \sigma^{\varepsilon,\theta}(t))(u^1(t) - u^0(t))\Big] dW(t),\ t \in [0,T], \end{aligned} \tag{5.25}$$

with initial condition $Y^{\varepsilon,\theta}(t) = 0,\ t \in [-\delta, 0]$.

By formally differentiating with respect to θ in (5.24), we obtain an equation of the form (5.25), suggesting that $\frac{d}{d\theta}X^{\varepsilon,\theta}(t) = Y^{\varepsilon,\theta}(t)$. This can be done rigorously:

$$\begin{aligned}\lim_{\theta\searrow\theta_0} \mathbb{E} \sup_{t\in[0,T]} \Bigg[& \left|\frac{X^{\varepsilon,\theta}(t) - X^{\varepsilon,\theta_0}(t)}{\theta-\theta_0} - Y^{\varepsilon,\theta_0}(t)\right|^2 \\ & + \left|\frac{K^{\varepsilon,\theta}(t) - K^{\varepsilon,\theta_0}(t)}{\theta-\theta_0} - \int_0^t \beta'_\varepsilon(X^{\varepsilon,\theta_0}(s))Y^{\varepsilon,\theta_0}(s)ds\right|^2\Bigg] = 0;\end{aligned} \tag{5.26}$$

for every $\theta_0 \in [0,T)$.

Our first task is to find an analogous derivative formula for X^θ and K^θ. For that, we consider the following linear equation:

$$\begin{aligned}dY^\theta(t) + Y^\theta(t)dA^\theta(t) = & \Big[\langle \partial_y b^\theta(t), R(Y^\theta)(t)\rangle + \langle \partial_u b^\theta(t), u^1(t) - u^0(t)\rangle\Big]\, dt \\ & + \Big[(\partial_y \sigma^\theta(t))R(Y^\theta)(t) + (\partial_u\sigma^\theta(t))(u^1(t) - u^0(t))\Big]\, dW(t),\ t\in[0,T],\end{aligned} \tag{5.27}$$

with initial condition $Y_0^\theta = 0,\ t \in [-\delta, 0]$.

Proposition 5.2.3. *Equation (5.27) has a unique solution* $Y^\theta \in L^2_{\mathbb{F}}(\Omega; C[-\delta,T])$.

Since the convergence in formula (5.26) is not necessarily uniform in $\varepsilon > 0$, we cannot derive a similar relation for X^θ, K^θ and Y^θ directly from that. In this regard, we will adapt an idea from (15) concerning the Malliavin derivatives for processes without control and we will define the derivative of $\theta \to X^\theta$ in a Sobolev space.

Proposition 5.2.4. *If* u^0 *and* u^1 *are càdlàg, the following derivation formula holds:*

$$\lim_{\theta\to 0}\mathbb{E}\left[\int_0^T \left|\frac{X^\theta(t) - X^0(t)}{\theta} - Y^0(t)\right|^2 dt + \left|\frac{X^\theta(T) - X^0(T)}{\theta} - Y^0(T)\right|^2\right] = 0. \tag{5.28}$$

The remaining part of this section is dedicated to the proof of Proposition5.2.4. For the moment we impose some restrictive assumptions:

(H_{16}) φ is affine outside a compact interval;

(H_{17}) there exists $c > 0$ such that $|\sigma(t,y,u)| \geq c$, for every $(t,y,u) \in [0,T]\times\mathbb{R}^d\times U$.

First, we need a stability result for A^u with respect to the control u:

Proposition 5.2.5. *Let $(u^n)_{n\geq 0}$ be a sequence of controls such that*

$$\sup_{t\in[0,T]} \left|u^n(t) - u^0(t)\right|^2 \to 0 \text{ in } L^\infty(\Omega).$$

Suppose that conditions (H_{16})-(H_{17}) hold and u^0 is càdlàg. Then

$$\mathbb{E}\left|A^{u^n}(t) - A^{u^0}(t)\right|^4 \to 0,\ \forall t \in [0,T]\,.$$

Let $\mathcal{H} := L^2_{\mathbb{F}}(\Omega \times [-\delta, T], P \otimes m_T)$, where the measure $m_T(dx) := dx + \delta_T(dx)$. We introduce the space $W^{1,2}([0,1]\,;\mathcal{H})$ of absolutely continuous functions (hence a.e. derivable) defined on $[0,1]$ and $\mathcal{H}$-valued.

Lemma 5.2.1 ((5))**.** *If the sequence $(X^n)_{n\geq 1} \subseteq W^{1,2}([0,1]\,;\mathcal{H})$ is bounded and converges in $L^2([0,1]\,;\mathcal{H})$ to some $X \in L^2([0,1]\,;\mathcal{H})$, then $X \in W^{1,2}([0,1]\,;\mathcal{H})$ and (∇X_n) converges weakly to ∇X in $L^2([0,1]\,;\mathcal{H})$.*

Relation (5.26) shows that $X^\varepsilon := (X^{\varepsilon,\theta})_{\theta\in[0,1]}$ and $K^\varepsilon := (K^{\varepsilon,\theta})_{\theta\in[0,1]}$ are elements of $W^{1,2}([0,1]\,;\mathcal{H})$; moreover, $\nabla_\theta X^\varepsilon = Y^{\varepsilon,\theta}$ and

$$\nabla_\theta K^\varepsilon = \int_0^\cdot \beta'_\varepsilon(X^{\varepsilon,\theta}(s))(\nabla_\theta X^\varepsilon)(s)ds,\ \theta \in [0,1]\,. \tag{5.29}$$

Also, X^ε and K^ε converge in $L^2([0,1]\,;\mathcal{H})$ to $X := (X^\theta)_{\theta\in[0,1]}$ and $K := (K^\theta)_{\theta\in[0,1]}$, respectively. By Lemma 5.2.1, it follows that ∇X^ε and ∇K^ε converge weakly in $L^2([0,1]\,;\mathcal{H})$ to ∇X, respectively ∇K.

By passing to the limit in equation (5.25) and using some *a priori* estimates, we have the following preliminary result:

Lemma 5.2.2. *We have $m_T(dt)dPd\theta$ a.e.*

$$\begin{aligned}(\nabla_\theta X)(t) + (\nabla_\theta K)(t) = &\int_0^t \left[\langle \partial_y b^\theta(s), R(\nabla_\theta X)(s)\rangle + \left\langle \partial_u b^\theta(s), u^1(t) - u^0(t)\right\rangle\right] ds \\ &+ \int_0^t \left[(\partial_y \sigma^\theta(s))R(\nabla_\theta X)(s) + (\partial_u \sigma^\theta(s))(u^1(t) - u^0(t))\right] dW(s).\end{aligned} \tag{5.30}$$

Moreover, under conditions (H_{16})-(H_{17}), $\nabla_\theta X$ can be chosen to be càdlàg, $\nabla_\theta K$ with bounded variation and càdlàg, satisfying

$$\operatorname*{esssup}_{\theta\in[0,1]} \mathbb{E}\left[\|\nabla_\theta X\|^4_{0,T} + \|\nabla_\theta K\|^4_{BV[0,T]}\right] < +\infty.$$

Lemma 5.2.3. *Suppose that u^0 and u^1 are càdlàg. Under (H_{16})-(H_{17}), the derivative of the mapping $\theta \longmapsto K^\theta$ in $W^{1,2}([0,1]\,;\mathcal{H})$ is given by:*

$$(\nabla_\theta K)(t) = \int_0^t (\nabla_\theta X)(s) dA^\theta(s),\ m_T(dt)dPd\theta \ a.e. \tag{5.31}$$

Relations (5.30) and (5.31), combined with the uniqueness of the solution of equation (5.27), give

$$(\nabla_\theta X)(t) = Y^\theta(t),\ \forall t \in [0,T]\,,\ dPd\theta \text{ a.e.} \tag{5.32}$$

5.2.3 Maximum principle for near optimal controls

We introduce the space of solutions as $\mathcal{S} := L^2_{\mathbb{F}}(\Omega; C[0,T]) \times L^2_{\mathbb{F}}(\Omega \times [0,T]; \mathbb{R}^d)$ and we define the Hamiltonian of the system $H : [0,T] \times \mathbb{R}^d \times U \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ by

$$H(t,y,u,p,q) = g(t,y,u) + b(t,y,u)p + \langle \sigma(t,y,u), q \rangle$$

For every control u, we consider the following anticipated BSDE on $[0,T]$:

$$\begin{cases} -\,dp(t) + p(t)dA^u(t) = \mathbb{E}^{\mathcal{F}_t}[F(t, R(X^u)(t), u(t), p(t), q(t))]dt \\ \qquad\qquad\qquad\qquad\qquad - \langle q(t), dW(t) \rangle\,; \\ p(T) = h'(X^u(T)), \end{cases} \tag{5.33}$$

where[1]

$$\begin{aligned} F(t,y,u,p,q) :=& \frac{\partial H}{\partial x}(t, y(t), p(t), q(t)) \\ &+ \int_t^{t+\delta} \frac{\partial H}{\partial y}(s, y(s), u(s), p(s), q(s)) \mathbf{1}_{[0,T]}(s) \lambda(t - ds) \end{aligned}$$

for $(t,y,u,p,q) \in [0,T] \times C([0,T]\,;\mathbb{R}^d) \times L^0([0,T];U) \times C\,[0,T] \times L^0([0,T];\mathbb{R}^d)$.

Theorem 5.2.2. *Equation (5.33) has a unique solution $(p,q) \in \mathcal{S}$.*

Every control u can be approximated, in $L^2(\Omega \times [0,T])$ by continuous controls u^ε. Hence, if u^* is an optimal control, since $J : L^2_{\mathbb{G}}(\Omega \times [0,T]) \to \mathbb{R}$ is continuous by Proposition 5.2.1, we can find continuous controls $\bar{u}^n$ with $\bar{u}^n \to u^*$ in $L^2(\Omega \times [0,T])$ and $J(\bar{u}^n) \le J(u^*) + n^{-1}$.

[1] $\mathbb{E}^{\mathcal{G}}\xi$ denotes the conditional expectation of a random variable ξ with respect to a subalgebra $\mathcal{G}$ of $\mathcal{F}$.

In order to apply Ekeland's variational principle, we take $\mathbb{X} := L^2_{\mathbb{G}}(\Omega; C([0,T];U))$, $\varepsilon = n^{-1}$ and $x = \bar{u}^n$. Therefore, for every $n \in \mathbb{N}^*$, there exist $u^n \in L^2_{\mathbb{G}}(\Omega; C([0,T];U))$ such that

$$\|u^n - \bar{u}^n\|_{L^2(\Omega; C([0,T];U))} \leq n^{-1/2}$$

and

$$J(u^n) \leq J^n(u) := J(u) + n^{-1/2} \|u - u^n\|_{L^2(\Omega; C([0,T];U))}, \ \forall u \in L^2_{\mathbb{G}}(\Omega; C([0,T];U)),$$

meaning that u^n is an optimal control corresponding to the perturbed cost functional J^n.

We now formulate the maximum principle for the near optimal controls u^n. Let $X^n := X^{u^n}$, $A^n := A^{u^n}$ and (p^n, q^n) be the solution of equation (5.33) with parameter u^n.

Proposition 5.2.6. *For every admissible control v we have*

$$\mathbb{E}\int_0^T \Big\langle \frac{\partial H}{\partial u}(t, R(X^n)(t), u^n(t), p^n(t), q^n(t)), v(t) - u^n(t) \Big\rangle dt \geq -\sqrt{\frac{1}{n}\mathbb{E}\sup_{t\in[0,T]} |v(t) - u^n(t)|^2}. \tag{5.34}$$

5.2.4 Maximum principle

We are able to retrieve the necessary conditions of optimality for u^* by passing to the limit in inequality (5.34). Let X^* denote the state of the system corresponding to the optimal control u^*.

Theorem 5.2.3 (maximum principle)**.** *If u^* is an optimal control, then there exists a càdlàg, bounded variation process $(K(t))_{t\in[0,T]}$ such that*

$$\Big\langle p^*(t)\frac{\partial b}{\partial u}(t, R(X^*)(t), u^*(t)) + q^*(t)\frac{\partial \sigma}{\partial u}(t, R(X^*)(t), u^*(t)) + \frac{\partial g}{\partial u}(t, R(X^*)(t), u^*(t)), v - u^*(t) \Big\rangle \geq 0,$$

$\forall v \in U$, $dtdP a.e.$,
where $(p^*, q^*) \in L^2_{\mathbb{F}}(\Omega \times [0,T]) \times L^2_{\mathbb{F}}(\Omega \times [0,T]; \mathbb{R}^d)$ is a solution of the equation

$$\begin{cases} -dp^*(t) = -dK(t) + \mathbb{E}^{\mathcal{F}_t}\left[F(t, R(X^*)(t), u^*(t), p^*(t), q^*(t))\right] dt - \langle q^*(t), dW(t)\rangle; \\ p^*(T) = h'(X^*(T)). \end{cases} \tag{5.35}$$

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